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Doubly Special Relativity and Finsler geometry

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Abstract

We discuss the recent proposal of implementing Doubly Special Relativity in configuration space by means of Finsler geometry. Although this formalism leads to a consistent description of the dynamics of a particle, it does not seem to give a complete description of the physics. In particular, the Finsler line element is not invariant under the deformed Lorentz transformations of Doubly Special Relativity. We study in detail some simple applications of the formalism.

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1. Introduction

In a recent paper [1] a relation between modified dispersion relations and Finsler geometry has been proposed, and in particular it has been observed that Doubly (or Deformed) Special Relativity (DSR) [2-4] can be realized in ordinary (commutative) configuration space as a (mass-dependent) Finsler geometry.

We recall that Finsler geometry is a generalization of Riemann geometry whose metric can depend both on position and velocity [5]. DSR models instead postulate a deformation of the standard Poincaré invariance of special relativity such that the momenta transform non-linearly under boosts, leaving invariant a fundamental energy scale κ (usually identified with the Planck energy). Such deformation can be obtained by suitably modifying the generators of boosts, and is not unique. The implementation of DSR in configuration space is not obvious, and has been extensively debated [6-9]. It is clear, however, that coordinate transformations consistent with DSR must be momentum dependent, and hence the possibility of a momentum (or velocity)-dependent geometry emerges.

This line of thought has been pursued in ref. [1]. However, although the formalism proposed there yields geodesics equations that transform covariantly under the deformed Lorentz transformations (DLT) of DSR, the Finsler line element is not invariant. As a consequence, it is not possible to define an invariant separation between events and it is difficult to identify a physical proper time in this framework. Moreover, it does not seem that the Finsler line element be the most natural affine parameter in this formalism.

In the present paper, we discuss these difficulties and give some explicit examples in the case of the best known DSR models.

DLT are a deformation of Poincaré algebra that do not alter the commutation relations between rotation generators M_i and boosts generators N_i ,^{*}

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, N_j\} = \epsilon_{ijk} N_k, \quad \{N_i, N_j\} = -\epsilon_{ijk} M_k, \quad (1.1)$$

and between rotations and translations generators p_a ,

$$\{M_i, p_0\} = 0, \quad \{M_i, p_j\} = \epsilon_{ijk} p_k, \quad (1.2)$$

but modify the commutation relations between boosts and translations generators:

$$\{N_i, p_a\} = w_a^i(p), \quad (1.3)$$

where the w_a^i are nonlinear functions of the momentum p and a dimensional parameter κ , usually identified with the Planck energy. The dispersion relation of the momentum are defined by means of the Casimir invariant $C(p)$ of the deformed algebra,

$$C(p) = m^2, \quad (1.4)$$

with m the mass of the particle.

^{*} We denote spacetime indices by a, b, \dots and spatial indices by i, j, \dots

According to [1], the Hamilton equations for a free particle in canonical phase space can be obtained from the reparametrization-invariant action

$$I = \int \left[\dot{q}_a p_a - \frac{\lambda}{2m} (C(p) - m^2) \right] d\tau, \quad (1.5)$$

where q are the coordinates of the particle, τ is an evolution parameter which is assumed to be invariant under DLT, and a dot denotes derivative with respect to it. The Lagrange multiplier λ enforces the dispersion relation (1.4), and the summations are performed with respect to the flat metric $\eta_{ab} = (1, -1, -1, -1)$. Varying with respect to q_a and p_a , one obtains

$$\dot{q}_a = \frac{\lambda}{m} \frac{\partial C}{\partial p_a}, \quad \dot{p}_a = 0. \quad (1.6)$$

To write the action in configuration space, one can invert the first equation, obtaining $p_a = p_a(m\dot{q}/\lambda)$. Then, substituting in the dispersion relation, one can obtain λ as a homogeneous function of \dot{q}_a of degree one. Finally, substituting back in the action,

$$I = \int \dot{q}_a p_a(\dot{q}) d\tau \equiv \int L(\dot{q}) d\tau. \quad (1.7)$$

The Lagrangian L depends only on the 4-velocity \dot{q} and is homogeneous of degree one in \dot{q} . One can therefore identify L with a Finsler norm. The Finsler line element is then defined as [5]

$$ds^2 = \frac{1}{2} \frac{\partial^2 L^2}{\partial q_a \partial q_b} dq_a dq_b, \quad (1.8)$$

and satisfies

$$ds = L d\tau, \quad (1.9)$$

by virtue of the homogeneity property of the Lagrangian.

It must be noted however that L , and hence ds , is not invariant under DLT. In fact, from the Jacobi identities one can deduce the infinitesimal transformation law of the coordinates q_a [7]. Indeed,

$$\{\{N_i, q_a\}, p_b\} + \{\{p_b, N_i\}, q_a\} + \{\{q_a, p_b\}, N_i\} = 0. \quad (1.10)$$

Assuming canonical Poisson brackets between phase space variables,

$$\{q_a, q_b\} = 0, \quad \{p_a, p_b\} = 0, \quad \{q_a, p_b\} = \eta_{ab}, \quad (1.11)$$

the last term of (1.10) vanishes, and one gets after integration

$$\{N_i, q_a\} = -\frac{\partial w_b^i}{\partial p_a} q_b. \quad (1.12)$$

It is then evident that the variation of the Finsler norm does not vanish in general. We can write in fact

$$\delta_i L(\dot{q}) = \delta_i(\dot{q}_a p_a) = \frac{d}{d\tau} [\delta_i(q_a p_a)] - \delta_i(q_a \dot{p}_a) = \frac{d}{d\tau} \left(w_a^i q_a - p_b \frac{\partial w_a^i}{\partial p_b} q_a \right), \quad (1.13)$$

where the term $\delta_i(q_a \dot{p}_a)$ vanishes as a consequence of (1.3) and (1.12). The remaining term can vanish only if the w_a^i are homogeneous function of degree one of the p , which rules out the standard DSR models, where a dimensionful parameter κ enters in the definition of the w_a^i . Therefore, also the line element ds is not invariant in general. The covariance of the geodesics equations obtained by varying (1.7) however persists since $\delta_i L$ is a total derivative.

One may notice that in the present formalism the natural definition of affine parameter seems not to coincide with the Finsler line element $ds = L d\tau$, but rather with $d\sigma \equiv \lambda d\tau$. In terms of σ , the Hamilton equations can in fact be written in the usual form

$$\frac{dq_a}{d\sigma} \equiv q'_a = \frac{1}{m} \frac{\partial C}{\partial p_a}, \quad \frac{dp_a}{d\sigma} \equiv p'_a = 0. \quad (1.14)$$

Moreover, being λ a homogeneous function $f(\dot{q})$ of degree one, one has

$$\lambda = f(\dot{q}) = \lambda f(q'), \quad (1.15)$$

from which one obtains a constraint on the four-velocity expressed in terms of the proper time, $f(q') = 1$, analogous to the relation $q_0'^2 - q_i'^2 = 1$ of special relativity.

The corresponding equations in terms of the Finsler line element take a much more involved form. Unfortunately, however, in general also the affine parameter σ is not invariant under DLT.

2. The Magueijo-Smolín model

In order to better understand the implications of the previous considerations, it is useful to consider some simple examples. The simplest one is the Magueijo-Smolín (MS) model [4], whose Lagrangian formulation has been studied in [10]. The dispersion relation is

$$\frac{p_0^2 - p_i^2}{(1 - p_0/\kappa)^2} = m^2. \quad (2.1)$$

which is left invariant by the DLT

$$\{N_i, p_0\} = p_i \left(1 - \frac{p_0}{\kappa}\right), \quad \{N_i, p_j\} = \delta_{ij} p_0 - \frac{1}{\kappa} p_i p_j. \quad (2.2)$$

From (1.10) follows [7]

$$\{N_i, q_0\} = q_i + \frac{p_i}{\kappa} q_0, \quad \{N_i, q_j\} = \left(1 - \frac{p_0}{\kappa}\right) q_0 \delta_{ij} + \frac{1}{\kappa} (p_k q_k \delta_{ij} + p_i q_j). \quad (2.3)$$

As already noted in [4], the deformed algebra is generated by the standard rotation generators, while the boost generators are given by

$$N_i = q_0 p_i - q_i p_0 - \frac{p_i}{\kappa} q_a p_a. \quad (2.4)$$

In order to simplify the calculation of the action, it is useful to write it as [10]

$$I = \int \left[\dot{q}_a p_a - \frac{\lambda}{2m} \left(p_a^2 - m^2 \left(1 - \frac{p_0}{\kappa} \right)^2 \right) \right] d\tau. \quad (2.5)$$

The field equations then read

$$\dot{q}_0 = \frac{\lambda}{m} \left(\frac{\kappa^2 - m^2}{\kappa^2} p_0 + \frac{m^2}{\kappa} \right), \quad \dot{q}_i = \frac{\lambda}{m} p_i, \quad (2.6)$$

and can be inverted explicitly,

$$p_0 = \frac{\kappa^2 m}{\kappa^2 - m^2} \left(\frac{\dot{q}_0}{\lambda} - \frac{m}{\kappa} \right), \quad p_i = m \frac{\dot{q}_i}{\lambda}. \quad (2.7)$$

Substituting into (2.1) one can obtain λ as a function of \dot{q} ,

$$\lambda = \sqrt{\dot{q}_0^2 - \frac{\kappa^2 - m^2}{\kappa^2} \dot{q}_i^2}. \quad (2.8)$$

The action can then be written in terms of the \dot{q} as

$$I = \int \dot{q}_a p_a(\dot{q}) d\tau = \frac{\kappa^2 m}{\kappa^2 - m^2} \int \left(\lambda - \frac{m}{\kappa} \dot{q}_0 \right) d\tau. \quad (2.9)$$

Notice that the last term is a total derivative and does not contribute to the field equations, that read

$$\frac{d}{d\tau} \left(\frac{\dot{q}_a}{\lambda} \right) = 0, \quad (2.10)$$

or, in terms of the affine parameter σ defined in section 1, $q_a'' = 0$, as in special relativity. However, now the 4-velocity satisfies the constraint

$$q_0'^2 - \frac{\kappa^2 - m^2}{\kappa^2} q_i'^2 = 1. \quad (2.11)$$

In terms of the parameter σ , the Lagrangian reads

$$L = \frac{\kappa^2 m}{\kappa^2 - m^2} \lambda \left(1 - \frac{m}{\kappa} q_0' \right). \quad (2.12)$$

Using the Finsler parameter s the equations would take a much more involved form.

We want now to investigate the transformation properties of the action. Under the action of a boost N_i , by (2.2)-(2.3),

$$\delta_i L = \delta_i(\dot{q}_a p_a) = \frac{1}{\kappa} \frac{d}{d\tau} (p_i p_a q_a). \quad (2.13)$$

In order to calculate the variation of λ , we consider for simplicity the two-dimensional case, where one has a single boost generator and the transformations (2.2)-(2.3) reduce to:

$$\delta p_0 = p_1 - \frac{p_0 p_1}{\kappa}, \quad \delta p_1 = p_0 - \frac{p_1^2}{\kappa}, \quad (2.14)$$

$$\delta q_0 = q_1 + \frac{p_1}{\kappa} q_0, \quad \delta q_1 = q_0 - \frac{p_0}{\kappa} q_0 + \frac{p_1}{\kappa} q_1. \quad (2.15)$$

Under these transformations, also the variation of λ reduces to a total derivative,

$$\delta \lambda = \frac{1}{\kappa} \frac{d}{d\tau} \left[q_1 + \frac{p_1}{\kappa} q_0 + \frac{\kappa^2 - m^2}{\kappa^2 m^2} p_1 (p_a q_a) \right]. \quad (2.16)$$

On shell, the variation of L and λ take the simpler form

$$\delta L = \frac{\kappa m^2 \dot{q}_1}{\kappa^2 - m^2} \left(1 - \frac{m \dot{q}_0}{\kappa \lambda} \right), \quad \delta \lambda = -\frac{2m \dot{q}_1}{\kappa}. \quad (2.17)$$

Thus, although the equations of motion are covariant under the DLT, neither the lagrangian L nor the Lagrange multiplier λ are invariant, but their variation is a total derivative. This of course is a problem if we wish to define an invariant proper time.

3. The Lukierski-Nowicki-Ruegg model

Another important example is given by the κ -Poincaré algebra of Lukierski-Nowicki-Ruegg [3]. In this case

$$\{N_i, p_0\} = p_i, \quad \{N_i, p_j\} = \frac{\kappa}{2} \left(1 - e^{-2p_0/\kappa} \right) \delta_{ij} + \frac{1}{2\kappa} (p_k p_k \delta_{ij} - 2p_i p_j), \quad (3.1)$$

and hence [7]

$$\{N_i, q_0\} = e^{-2p_0/\kappa} q_i, \quad \{N_i, q_j\} = q_0 \delta_{ij} + \frac{1}{\kappa} (p_k q_k \delta_{ij} + p_i q_j - p_j q_i). \quad (3.2)$$

The dispersion relation invariant under (3.1)-(3.2) is

$$C(p) = 4\kappa^2 \sinh^2(p_0/2\kappa) - e^{p_0/\kappa} p_i^2 = m^2 \quad (3.3)$$

It is then easy to write down the deformed generators of the boosts

$$N_i = p_i q_0 - \frac{\kappa}{2} \left(1 - e^{-2p_0/\kappa} \right) q_i - \frac{1}{2\kappa} (p_k p_k q_i - 2p_i p_k q_k). \quad (3.4)$$

Varying the action (1.4) one obtains the Hamilton equations

$$\dot{q}_0 = \frac{\lambda}{m} \left[\sinh(p_0/\kappa) - e^{p_0/\kappa} \frac{p_i^2}{2\kappa} \right], \quad \dot{q}_i = \frac{\lambda}{m} e^{p_0/\kappa} p_i. \quad (3.5)$$

In this case it is not possible to invert explicitly the equations for p in terms of \dot{q} , so we expand them in powers of m/κ ,

$$p_0 \sim m \left(\frac{\dot{q}_0}{\lambda} + \frac{m}{2\kappa} \frac{\dot{q}_i^2}{\lambda^2} \right), \quad p_i \sim m \left(\frac{\dot{q}_i}{\lambda} - \frac{m}{\kappa} \frac{\dot{q}_0 \dot{q}_i}{\lambda^2} \right). \quad (3.6)$$

Substituting in (3.3),

$$\lambda \sim \sqrt{\dot{q}_0^2 - \dot{q}_i^2} + \frac{m}{\kappa} \frac{\dot{q}_0 \dot{q}_i^2}{\dot{q}_0^2 - \dot{q}_i^2}, \quad (3.7)$$

and finally,

$$I \sim m \int \left[\sqrt{\dot{q}_0^2 - \dot{q}_i^2} + \frac{m}{2\kappa} \frac{\dot{q}_0 \dot{q}_i^2}{\dot{q}_0^2 - \dot{q}_i^2} \right] d\tau. \quad (3.8)$$

In terms of the affine parameter σ ,

$$L \sim m\lambda \left(1 - \frac{m}{2\kappa} q'_0 q_i'^2 \right). \quad (3.9)$$

Note that, contrary to the MS case, now L and λ do not differ by a total derivative.

To consider the effect of the transformations (3.1)-(3.2) on the Lagrangian, we again consider the two-dimensional case, for which

$$\delta p_0 = p_1, \quad \delta p_1 = \frac{\kappa}{2} \left(1 - e^{-2p_0/\kappa} \right) + \frac{p_1^2}{2\kappa} \quad (3.10)$$

$$\delta q_0 = e^{-2p_0/\kappa} q_1, \quad \delta q_1 = q_0 + \frac{p_1}{\kappa} q_1. \quad (3.11)$$

Under these transformations, the Lagrangian changes by a total derivative (see 2.13),

$$\delta L = \frac{d}{dt} \left[\left(p_0 - \frac{\kappa}{2} (1 - e^{-2p_0/\kappa}) - \frac{p_1^2}{2\kappa} \right) q_1 \right] \sim -\frac{1}{2\kappa} \frac{d}{d\tau} [(2p_0^2 + p_1^2) q_1]. \quad (3.12)$$

The transformation properties of λ are instead quite involved and its variation under DLT is not even a total derivative.

4. Conclusions

As we have shown, the formalism of ref. [1] permits to write down DLT-covariant equations for the geodesic motion of a point particle in canonical configuration space of DSR. However, it does not allow to define an invariant affine parameter, and hence to identify the physical proper time.

The situation does not change by passing to noncommutative spacetime coordinates. In fact in this case, in order to obtain the correct Hamilton equations, one has to modify the term $p\dot{q}$ in the action [7], and the new term is still not invariant under the DLT compatible with the new symplectic structure.

It seems therefore that although Finsler spaces are useful for studying the motion of DSR particles, they do not catch the full structure of the theory. This is presumably

due to the fact that Finsler spaces are tailored for the study of homogeneous dispersion relations, while DSR dispersion relations cannot be homogeneous because of the presence of a dimensional constant κ . Moreover, the formalism of Finsler spaces is built in such a way to avoid the presence of Lagrangian multipliers in the action principle [11].

A possible solution might be to consider some generalizations of Finsler spaces, either defining a metric structure in the full phase space, instead of configuration space, or perhaps by relaxing the requirement of homogeneity of the Finsler metric. A last possibility is that the formalism work better when a five-dimensional configuration space is considered as in [9,12].

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References

- [1] F. Girelli, S. Liberati and L. Sindoni, Phys. Rev. **D75**, 064015 (2007).
- [2] G. Amelino-Camelia, Int. J. Mod. Phys. **D11**, 35 (2002), Phys. Lett. **B510**, 255 (2001).
- [3] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. **B264**, 331 (1991); J. Lukierski, A. Nowicki and H. Ruegg, Phys. Lett. **B293**, 344 (1992).
- [4] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002).
- [5] H. Rund *The differential geometry of Finsler spaces*, Springer-Verlag 1959.
- [6] J. Kowalski-Glikman, Mod. Phys. Lett. **A17**, 1 (2002).
- [7] S. Mignemi, Phys. Rev. **D68**, 065029 (2003); Phys. Rev. **D72**, 087703 (2005).
- [8] D. Kimberly, J. Magueijo and J. Medeiros, Phys. Rev. **D70**, 084007 (2004).
- [9] A.A. Deriglazov and B.F. Rizzuti, Phys. Rev. **D71**, 123515 (2005).
- [10] S. Ghosh, Phys. Rev. **D74**, 084019 (2006).
- [11] H. Rund *The Hamilton-Jacobi theory in the calculus of variations*, Krieger 1973.
- [12] F. Girelli, T. Konopka, J. Kowalski-Glikman and E.R. Livine, Phys. Rev. **D73**, 045009 (2006).